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Closure of macroscopic laws in disordered spin systems: a toy model

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Abstract. We use a linear system of Langevin spins with disordered interactions as an exactly solvable toy model to investigate a procedure, recently proposed by Coolen and Sherrington, for closing the hierarchy of macroscopic order parameter equations in disordered spin systems. The closure procedure, based on the removal of microscopic memory effects, is shown to reproduce the correct equations for short times and in equilibrium. For intermediate time-scales the procedure does not lead to the exact equations, yet for homogeneous initial conditions succeeds in capturing the main characteristics of the flow in the order parameter plane. The procedure fails in terms of the long-term temporal dependence of the order parameters. For low-energy inhomogeneous initial conditions and near criticality (where zero modes appear) deviations in temporal behaviour are most apparent. For homogeneous initial conditions the impact of microscopic memory effects on the evolution of macroscopic order parameters in disordered spin systems appears to be mainly an overall slowing down.

The off-equilibrium dynamics of mean-field spin–glass models has been a central subject of research in recent times [1–4]. Starting from a stochastic Markovian dynamics for the microscopic variables, it is found that the appropriate 'order parameter functions'—the spin–spin correlation function at different times, and the associated response function—obey non-Markovian equations. These contain, at all times, 'memory terms' which depend on the previous history. It has been emphasized recently that these memory terms play an important role at low temperatures, being responsible for asymptotic breaking of time translation invariance and aging.

If one wants to retain a description of the dynamics at a Markovian level, it is possible to write down an exact hierarchy of equations which does not close. An infinite number of instantaneous 'order parameters' is therefore required. Recently Coolen and Sherrington [7, 8] (CS) proposed a method for obtaining a closed set of autonomous macroscopic *differential equations*, *by a systematic elimination of microscopic memory effects*. The crucial simplifying hypothesis is that at any time-step the system is at equilibrium on the surface where some macroscopic variables related to the energy and the magetization are constant. They applied this method to the Hopfield [9] model and the Sherrington–Kirkpatrick (SK) [10] model, which are the archetypical models for attractor neural networks and spin–glasses, respectively. The CS procedure, by construction, gives the exact equations at $t = 0$ and the correct equilibrium fixed points. In addition it is exact for all times in the limit where the disorder is removed. For the SK model and the Hopfield model, however,

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there are no sufficiently detailed and reliable analytical or numerical results to allow for a direct test of the procedure at intermediate times, although recent data on cumulants of the local field distribution in the Hopfield model do suggest deviations between theory and simulations [11]. For the Hopfield model, for instance, non-trivial finite-size effects are known to persist even in systems of size $N \sim 10^6$ [12]. Secondly, in both models a full analysis following CS requires going through a dynamic version of the replica symmetry breaking scheme *à la* Parisi [13], for a continuous range of times and parameters, which is, in practice, unattainable.

Nevertheless, it is worth noticing that even if the CS procedure neglects the possibility of aging—the crucial aspect of the spin–glass dynamics—the 'flow diagrams' obtained for the order parameters in [7, 8] show qualitative agreement with the results of numerical simulations. In order to understand the potential and the restrictions of the closure procedure proposed by CS, we present in this paper results of studying an exactly solvable toy model, for which replica symmetry in the dynamical equations is stable. Since the CS procedure is based on the elimination of microscopic memory effects, this also allows us to obtain a better understanding of the role of these effects in determining the macroscopic behaviour of disordered spin systems.

As our toy model we choose a linear system of N Langevin spins $\{\sigma_i\}$ with disordered interactions $\{J_{ij}\}$ and a Gaussian white noise $\{\eta_i(t)\}$:

$$\frac{d}{dt}\sigma_i = \sum_{j=1}^{N} J_{ij}\sigma_j - \mu\sigma_i + \eta_i \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t') \tag{1}$$

in which the symmetric interactions $\{J_{ij}\}$ are drawn at random from a Gaussian distribution with $\langle J_{ij}\rangle = 0$ and $\langle J_{ij}^2\rangle = J^2N^{-1}$. For large N the eigenvalue distribution $\rho(\lambda)$ of the interaction matrix is given by Wigner's semi-circular law [14]

$$\rho(\lambda) = \frac{\sqrt{4J^2-\lambda^2}}{2\pi J^2}\theta[2J+\lambda]\,\theta[2J-\lambda] \tag{2}$$

so we have to choose $\mu \geqslant 2J$ in order to suppress runaway modes. Both the statics and the dynamics of this linear model are solved trivially. In statics, it is found that the replica-symmetric solution is always stable and the model does not have a glassy phase. The system is always in a 'high-temperature' condition. Here we can expect that the memory effects only play a minor role, compared to spin–glass systems which exhibit a transition. The dynamics are solved by transformation to the basis where the interaction matrix is diagonal, i.e. $\sigma_i(t) \to \sigma_\lambda(t)$ and $\eta_i(t) \to \eta_\lambda(t)$ (this transformation does not affect the statistical properties of the noise). Alternatively, one could write coupled equations for correlation and response functions, containing memory terms. Following the former route one obtains

$$\sigma_\lambda(t) = \sigma_\lambda(0)e^{-t(\mu-\lambda)} + \int_0^t ds\,\eta_\lambda(s)e^{(\mu-\lambda)(s-t)}. \tag{3}$$

In particular, we can calculate directly the quantitities in terms of which the CS procedure will be formulated, the average spin norm $Q \equiv (1/N)\sum_i \sigma_i^2$ and the energy per spin $E \equiv -(1/2N)\sum_{ij}\sigma_i J_{ij}\sigma_j + (\mu/2N)\sum_i \sigma_i^2$:

$$Q(t) = \int d\lambda\,\rho(\lambda)\sigma_\lambda^2(0)e^{-2t(\mu-\lambda)} + T\int d\lambda \frac{\rho(\lambda)}{\mu-\lambda}\left[1-e^{-2t(\mu-\lambda)}\right] \tag{4}$$

$$E(t) = \tfrac{1}{2}\int d\lambda\,\rho(\lambda)\sigma_\lambda^2(0)(\mu-\lambda)e^{-2t(\mu-\lambda)} + \tfrac{1}{2}T\left[1-\int d\lambda\,\rho(\lambda)e^{-2t(\mu-\lambda)}\right] \tag{5}$$

where $\sigma_\lambda^2(0)$ is the contribution per degree of freedom to $Q(0)$ from eigenspace λ, i.e. $Q(0) = (1/N)\sum_i \sigma_i^2(0) = \int d\lambda\, \rho(\lambda)\sigma_\lambda^2(0)$. We will (improperly) call the quantities Q and E 'order parameters' for the system. The macroscopic equilibrium state is found to be

$$Q(\infty) = \frac{T}{2J^2}\left[\mu - \sqrt{\mu^2 - 4J^2}\right] \qquad E(\infty) = \tfrac{1}{2}T\,. \tag{6}$$

We will consider two types of initial conditions. For homogeneous initial conditions, $\sigma_\lambda^2(0) = Q_0$, we can write the solution (4) and (5) in the compact form

$$Q(t) = Q_0 - 4\int_0^t ds\left[E(s) - \tfrac{1}{2}T\right] \tag{7}$$

$$E(t) = \tfrac{1}{2}T - \tfrac{1}{2}\left[\tfrac{1}{2}Q_0\frac{d}{dt} + T\right]\frac{e^{-2\mu t}I_1(4Jt)}{2Jt} \tag{8}$$

in which $I_1(z)$ denotes the modified Bessel function [15]. We can use the properties of $I_1(z)$ to obtain from (8) directly the short-time and the asymptotic behaviour of the system, for comparison with the results of the CS procedure. For short times we find

$$E(t) = \tfrac{1}{2}\mu Q_0 + t\left[\mu T - Q_0(\mu^2 + J^2)\right] - t^2\left[T(\mu^2 + J^2) - \mu Q_0(\mu^2 + 3J^2)\right]$$
$$+\tfrac{2}{3}t^3\left[\mu T(\mu^2 + 3J^2) - Q_0(\mu^4 + 6\mu^2J^2 + 2J^4)\right] + \mathcal{O}(t^4) \qquad t \to 0 \tag{9}$$

whereas the asymptotic behaviour turns out to be described by

$$E(t) = \tfrac{1}{2}T - \frac{e^{-2t(\mu-2J)}}{4\sqrt{\pi}(2Jt)^{3/2}}\left[T - Q_0(\mu - 2J) + \mathcal{O}\left(\frac{1}{t}\right)\right] \qquad t \to \infty\,. \tag{10}$$

The second type of initial conditions we consider are inhomogeneous ones, where the system is prepared in one specific eigendirection Λ of the interaction matrix, so $\sigma_\lambda^2(0) = Q_0\delta(\lambda - \Lambda)\rho^{-1}(\Lambda)$. The solution (4) and (5) can now be written as

$$Q(t) = Q_0 - 4\int_0^t ds\left[E(s) - \tfrac{1}{2}T\right] \tag{11}$$

$$E(t) = \tfrac{1}{2}Q_0(\mu - \Lambda)e^{-2t(\mu-\Lambda)} + \tfrac{1}{2}T\left[1 - \frac{e^{-2\mu t}I_1(4Jt)}{2Jt}\right]. \tag{12}$$

For short times we now find

$$E(t) = \tfrac{1}{2}(\mu - \Lambda)Q_0 + t\left[\mu T - Q_0(\mu - \Lambda)^2\right] - t^2\left[T(\mu^2 + J^2) - Q_0(\mu - \Lambda)^3\right]$$
$$+\tfrac{2}{3}t^3\left[\mu T(\mu^2 + 3J^2) - Q_0(\mu - \Lambda)^4\right] + \mathcal{O}(t^4) \qquad t \to 0 \tag{13}$$

whereas the asymptotic behaviour is given by

$$E(t) = \tfrac{1}{2}T + \tfrac{1}{2}(\mu - \Lambda)Q_0 e^{-2t(\mu-\Lambda)} - \frac{Te^{-2t(\mu-2J)}}{4\sqrt{\pi}(2Jt)^{3/2}}\left[1 + \mathcal{O}\left(\frac{1}{t}\right)\right] \qquad t \to \infty\,. \tag{14}$$

For $\mu > 2J$ there are no zero modes and the asymptotic relaxation is simply exponential, with characteristic time $\tau = [2(\mu - 2J)]^{-1}$. For $\mu = 2J$, however, zero modes appear, as a result of which we find power laws

$$Q(t) - Q(\infty) \sim t^{-1/2} \qquad E(t) - E(\infty) \sim t^{-3/2}\,. \tag{15}$$

The only asymptotic diffence between the two types of initial conditions is that in the case of choosing a zero mode as the initial state ($\mu = \Lambda = 2J$), the equilibrium norm $Q(\infty)$ will depend on Q_0.

We now turn to the CS procedure [7, 8] for deriving a closed set of deterministic macroscopic differential equations. The total energy per spin is separated into two contributions, one of which depends on the realization of the disorder, and one of which does not. These two quantities (or equivalently any functions thereof) will evolve in time deterministically on finite time-scales and are chosen to represent a macroscopic state. For the present model we can choose Q and E. From the Fokker–Planck equation associated with (1) follows a Liouville equation for the macroscopic probability distribution $\mathcal{P}_t(Q,E)$, which describes the deterministic flow

$$\frac{\mathrm{d}}{\mathrm{d}t}Q = -4\big[E - \tfrac{1}{2}T\big] \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \mu T - \langle h^2\rangle_{Q,E;t} \tag{17}$$

with the sub-shell average

$$\langle h^2\rangle_{Q,E;t} \equiv \frac{\int \mathrm{d}\sigma\, p_t(\sigma)\,\delta[Q-Q(\sigma)]\,\delta[E-E(\sigma)]\,(1/N)\sum_i\big[\sum_j J_{ij}\sigma_j - \mu\sigma_i\big]^2}{\int \mathrm{d}\sigma\, p_t(\sigma)\,\delta[Q-Q(\sigma)]\,\delta[E-E(\sigma)]}. \tag{18}$$

These laws are exact, although not yet closed due to the appearance of the microscopic probability distribution $p_t(\sigma)$ in (18). In the case of the Hopfield [9] and the Sherrington–Kirkpatrick [10] model, removing the disorder (by putting $\alpha = 0$ and $\tilde{J} = 0$ in these models, respectively) closes the hierarchy [7, 8]. The same happens in the present toy model: for $J = 0$ the sum over sites in (18) simply equals $\mu^2 Q$, the microscopic distribution $p_t(\sigma)$ drops out and (16) and (17) close.

Following CS we now close (16) and (17) for arbitrary J by assuming (i) self-averaging of the flow with respect to the microscopic realization of the disorder, and (ii) that in evaluating the disorder-averaged sub-shell average $\langle h^2\rangle_{Q,E;t}$ (18) we may assume equipartitioning of probability within the (Q,E) sub-shells of the ensemble. As a result $\langle h^2\rangle_{Q,E;t}$ is replaced by

$$\langle h^2\rangle_{Q,E} \equiv \left\langle \frac{\int \mathrm{d}\sigma\,\delta[Q-Q(\sigma)]\,\delta[E-E(\sigma)]\,(1/N)\sum_i\big[\sum_j J_{ij}\sigma_j - \mu\sigma_i\big]^2}{\int \mathrm{d}\sigma\,\delta[Q-Q(\sigma)]\,\delta[E-E(\sigma)]}\right\rangle_{\{J_{ij}\}}. \tag{19}$$

The set (16) and (17) is now closed and the sub-shell average (19) can be calculated with the replica method. From this stage onwards all calculations can be performed exactly. The underlying assumptions are guaranteed to be exact at $t = 0$ (upon choosing appropriate initial conditions) and in equilibrium (as a result of the Boltzmann form of the equilibrium distribution).

In calculating (19) with the replica method there enters an auxiliary spin–glass-type order parameter $q(Q,E)$, with the physical meaning

$$q \equiv \left\langle \frac{\int\!\int \mathrm{d}\sigma\,\mathrm{d}\sigma'\,\frac{1}{N}\sum_i \sigma_i\sigma'_i\,\delta[Q-Q(\sigma)]\delta[E-E(\sigma)]\delta[Q-Q(\sigma')]\delta[E-E(\sigma')]}{\int\!\int \mathrm{d}\sigma\,\mathrm{d}\sigma'\,\delta[Q-Q(\sigma)]\delta[E-E(\sigma)]\delta[Q-Q(\sigma')]\delta[E-E(\sigma')]}\right\rangle_{\{J_{ij}\}}. \tag{20}$$

Upon making the replica-symmetry (RS) ansatz one finds three different regions in the (Q,E) plane, characterized by different associated values of q and of the relevant sub-shell average $\langle h^2\rangle_{Q,E}$, the boundaries of which are the lines $E = E_{\mathrm{A}}$ and $E = E_{\mathrm{B}}$:

$$E_{\mathrm{A}} \equiv \tfrac{1}{2}Q(\mu - J) \qquad E_{\mathrm{B}} \equiv \tfrac{1}{2}Q(\mu + J). \tag{21}$$

By calculating the eigenvalue λ_{AT}, which determines the stability of the RS saddle point in the so-called replicon direction [16], we find that RS is truly stable in the $q = 0$ region

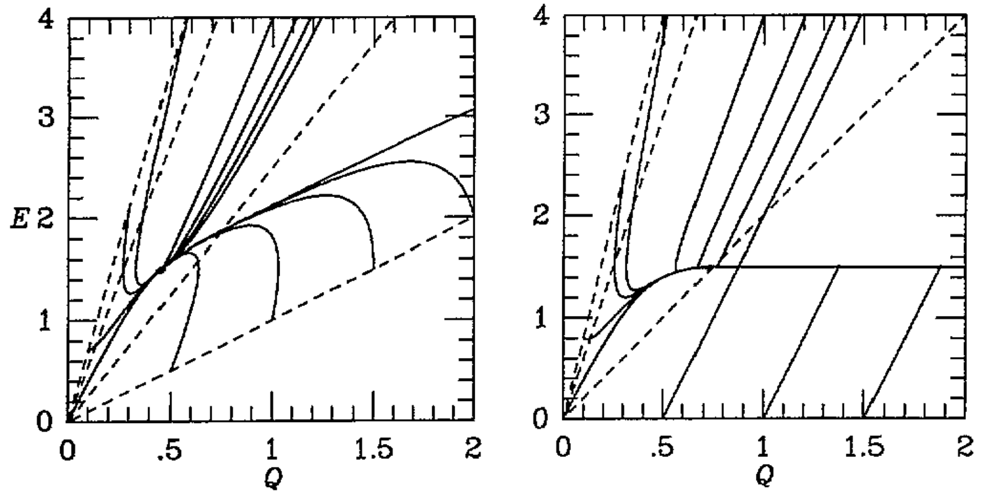

Figure 1. Flow in the (Q, E) plane according to the CS equations, for $\mu = 8$ and $T = 3$. Left-hand figure: $J/\mu = \frac{3}{8}$ (no zero modes), right-hand figure: $J/\mu = \frac{1}{2}$ (zero modes). Outer two broken lines: boundaries of the physical region, $E = \frac{1}{2}Q(\mu \pm 2J)$ (note, for $J/\mu = \frac{1}{2}$ one of these coincides with the line $E = 0$). Inner two broken lines: boundaries $E_{A,B}$ of the $q = 0$ region. Thin horizontal line segment in the right-hand figure represents the degenerated stable line $E = \frac{1}{2}T$.

($\lambda_{AT} > 0$), and marginally stable in the two $q > 0$ regions ($\lambda_{AT} = 0$):

Region	q	λ_{AT}	$\mathrm{d}E/\mathrm{d}t$
$E < E_A$	> 0	0	$-2\mu(E - \frac{1}{2}T) + (\mu - 2J)(\mu Q - 2E)$
$E_A < E < E_B$	0	> 0	$\mu T - QJ^2 - 4E^2/Q$
$E > E_B$	> 0	0	$-2\mu(E - \frac{1}{2}T) + (\mu + 2J)(\mu Q - 2E)$.

(22)

In the limit of zero disorder we indeed recover the correct (trivial) evolution for the remaining order parameter Q. We will now assess to what degree the CS flow equations (16) and (22) reproduce or approach the exact results in the presence of disorder.

According to (16) and (22) the flow in the two $q > 0$ regions is directed into the middle region $q = 0$, where the fixed-point of the flow is indeed given by the correct expression (6). At criticality ($\mu = 2J$), however, the $q = 0$ fixed-point is precisely on the regional boundary $E = E_A$ and a degenerated stable line $E = \frac{1}{2}T$ develops in the region $E < E_A$. Solving numerically the flow equations (16) and (22) results in figure 1 (in this example $\mu = 8$ and $T = 3$), where we show the flow iterated for $0 \leqslant t \leqslant 10$ from initial states which are drawn either homogeneously (on the line $E = \frac{1}{2}\mu Q_0$) or inhomogeneously from the extreme modes $\lambda = \pm 2J$ (on the lines $E = \frac{1}{2}Q_0(\mu \pm 2J)$, which are the boundaries of the physical region).

The corresponding flow according to the exact equations (8) and (12) is shown in figure 2. Away from the critical situation $\mu = 2J$ there is a qualitative agreement between the two flows, especially for homogeneous initial conditions. For $\mu = 2J$ there are clear deviations in the region $E < E_A$ (where one finds the zero modes). In this picture we have added flow lines starting from initial states with $E = \frac{1}{2}Q_0(\mu - 2J) + \epsilon$ ($\epsilon \ll 1$), to emphasize the difference with the zero modes $E = \frac{1}{2}Q_0(\mu - 2J)$. The degenerated line $E = \frac{1}{2}T$ of the CS equations is in reality not stable.

If we inspect the temporal behaviour away from the critical situation $\mu = 2J$, we find

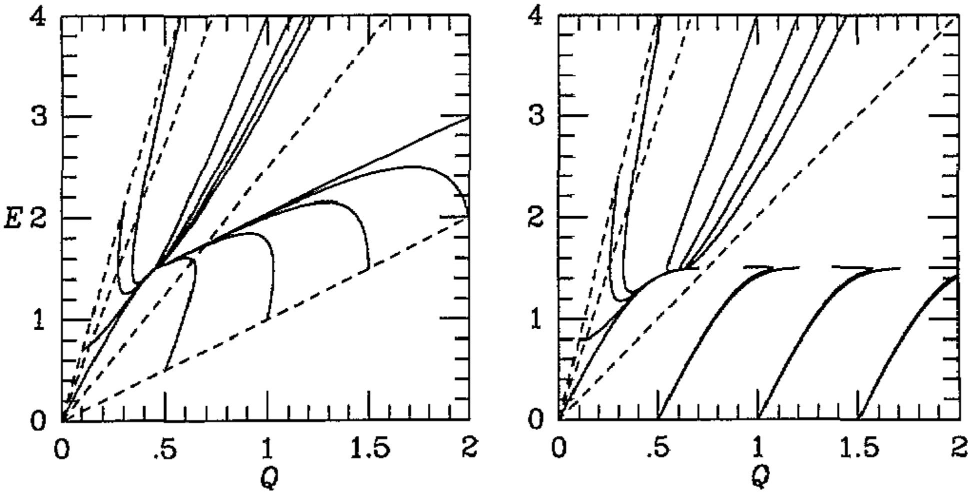

Figure 2. Flow in the (Q, E) plane according to the exact solution, for $\mu = 8$ and $T = 3$. Left-hand figure: $J/\mu = \frac{3}{8}$ (no zero modes), right-hand figure: $J/\mu = \frac{1}{2}$ (zero modes). Outer broken lines: boundaries of the physical region, $E = \frac{1}{2}Q(\mu \pm 2J)$ (note, for $J/\mu = \frac{1}{2}$ one of these coincides with the line $E = 0$). Inner broken lines: boundaries $E_{A,B}$ of the $q = 0$ region.

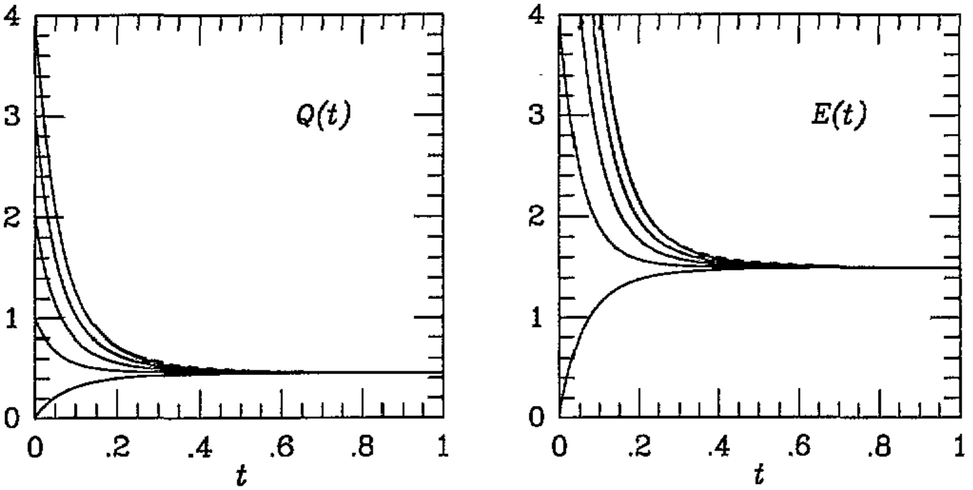

Figure 3. Comparison of order parameter evolution starting from homogeneous initial states, for $\mu = 8$, $T = 3$ and $J/\mu = \frac{3}{8}$ (no zero modes). Full curves, exact equations; broken curves, CS closure.

a remarkable agreement for the case of homogeneous initial conditions (see figure 3). In the case of inhomogeneous initial conditions (see figure 4) there is a difference between starting from high-energy initial states $\lambda = 2J$, where there is again agreement between exact and CS results, and starting from low-energy initial states $\lambda = -2J$, where significant deviations occur. Expansion of the flow equations (16) and (22) around the equilibrium state gives the leading asymptotic temporal behaviour, which can be compared to the exact

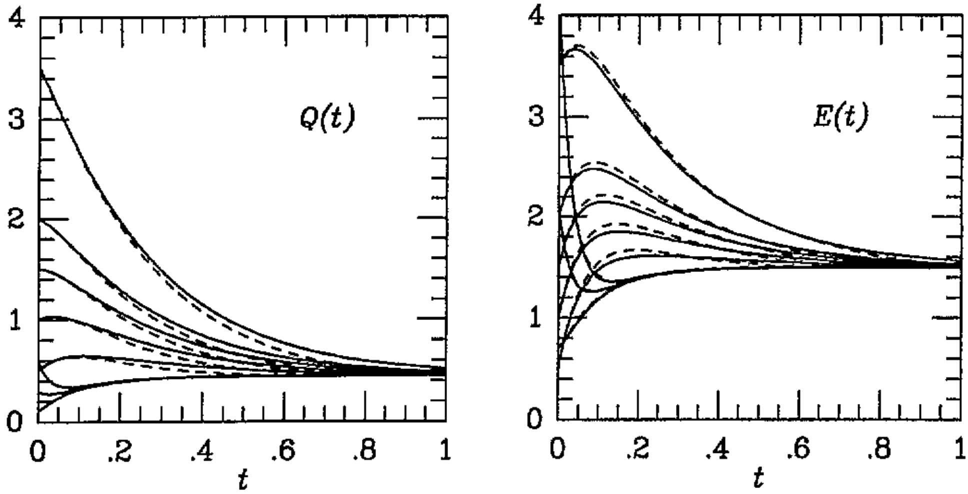

Figure 4. Comparison of order parameter evolution starting from inhomogeneous initial states, for $\mu = 8$, $T = 3$ and $J/\mu = \frac{3}{8}$ (no zero modes). Full curves, exact equations; broken curves, CS closure.

results (with the dimensionless quantity $x \equiv 2J/\mu$):

$$
\begin{array}{llll}
\mu > 2J: & E(t) - E(\infty) \sim e^{-t/\tau} & Q(t) - Q(\infty) \sim e^{-t/\tau} & \\
 & \text{exact:} & (2\mu\tau)^{-1} = 1 - x & \\
 & \text{CS:} & (2\mu\tau)^{-1} = \frac{1}{2}(1-x) + \frac{1}{2}\sqrt{1-x^2} & \\
\mu = 2J: & E(t) - E(\infty) \sim t^{-\alpha-1} & Q(t) - Q(\infty) \sim t^{-\alpha} & (23) \\
 & \text{exact:} & \alpha = \frac{1}{2} & \\
 & \text{CS:} & \alpha = 1\,. &
\end{array}
$$

The agreement obtained for homogeneous initial conditions, in spite of the difference in characteristic relaxation times (23), can be explained by studying the behaviour of $\log[Q - Q(\infty)]$ and $\log[E - E(\infty)]$ as a function of time for the exact solution (8). It turns out that the regime of exponential relaxation only sets in extremely close to equilibrium (for $\log[Q - Q(\infty)]$ and $\log[E - E(\infty)]$ of the order of 10^{-15}). For short times one can expand the CS equations in powers of t and compare with the exact expansions (9) and (13), with the following results:

$$
\begin{array}{ll}
E(0) = \frac{1}{2}\mu Q_0 & E_{\mathrm{CS}}(t) = E_{\mathrm{exact}}(t) + \mathcal{O}(t^3) \\
E(0) = \frac{1}{2}(\mu \pm 2J) Q_0 x & E_{\mathrm{CS}}(t) = E_{\mathrm{exact}}(t) + \mathcal{O}(t^2)
\end{array}
\qquad (24)
$$

which explains the difference in agreement between the two types of initial conditions, i.e. between figures 3 and 4.

The aim of our study was to perform a test of the closure procedure proposed in [7, 8], which is based on the removal of any memory effects in the evolution of macroscopic quantities. For such a test we have chosen a model which is exactly solvable, and does not involve the full hierarchy of replica-symmetry breaking in the dynamical calculations of CS. Our study shows that the CS procedure does not lead in any non-trivial case to the exact equations, yet in some cases succeeds in capturing the main characteristics of the flow in the order parameter plane. The procedure fails in terms of the long-term temporal dependence of the order parameters. However, in the absence of zero modes, and for homogeneous initial conditions the true asymptotic regime turns out to have only restricted relevance, as it sets in extremely late. This implies that for homogeneous initial conditions the effect of microscopic memory effects on the evolution of macroscopic order parameters results mainly in an overall slowing down.

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